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# A mechanism for instability of plane Couette flow and of Poiseuille flow in a pipe

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It is found that only a small change in either of the undisturbed velocity profiles concerned is required to change them from stable profiles to unstable profiles. The change must be such as to produce a local maximum in the magnitude of the vorticity, or in the case of the pipe, in the magnitude of the vorticity divided by the radius. The actual change in the vorticity (or vorticity/radius) need only be small, but the gradient of the vorticity (or vorticity/radius) must be finite. Viscosity will tend to damp out the distortion in the mean flow that is responsible for the instability, so that if the flow is to become turbulent, non-linear effects must become important before the distortion of the mean flow is reduced to an ineffective level. This requirement leads to the determination of critical Reynolds numbers which depend on the initial (small) distortion of the mean flow and the initial (smaller) amplitude of periodic disturbances. These critical Reynolds numbers are large.

## 1. Introduction

The motivation for this work comes from some experimental results of Leite (1956) reported also by Kuethe (1956). In these experiments, disturbances to the flow of air through a pipe were produced by oscillations of a circular airfoil. The experiments of particular relevance to this discussion were performed at a Reynolds number of 12,000, observations of axial velocity being made by means of radial hot-wire traverses at cross-sections approximately 3, 6, 10 and 47 diameters downstream of the airfoil. The remarks I wish to make are mainly concerned with the observations at the first three downstream stations, as the flow was practically fully turbulent at the station 47 diameters downstream of the airfoil. In the following discussion, the unit of velocity will be taken as twice the mean velocity over a cross-section of the pipe, which would be equal to the velocity at the centre of the pipe if the flow were strictly Poiseuille flow. With this unit, the deviation of the mean flow profile from Poiseuille flow at each of the first three stations was found to be a function of the radius with a maximum deviation of between $\frac{1}{15}$ and $\frac{1}{20}$. The deviation seemed to correspond to a wake behind the airfoil. In contrast, the amplitude of the periodic disturbance produced by the oscillations of the airfoil was only $\frac{1}{600}$ at the first station, but rose rapidly to a value of $\frac{1}{70}$ at the second station and $\frac{1}{14}$ (comparable with the mean flow deviation) at the third. This seems to demonstrate that the change in the mean flow was imposed by the airfoil and that the changed mean flow profile was unstable. Before starting the work reported here, I had one result available which explains the lack of change in the distortion of the mean flow in the first eight or so diameters where the Reynolds stress due to the periodic disturbance must have been too small to have any significant effect. This result (Gill 1965, §6) showed that the change in the mean flow could be expressed as the sum of a series of modes, each damped out at a different rate. At a Reynolds number of 12,000, however, the result shows that the first three modes would only be reduced by 5, 15 and 27 % in a downstream distance of 10 diameters.

## 2. Instability of distorted profiles

It remained, then, to show that the distorted profile is unstable and that the growth rate is large enough for non-linear effects to become important before the distortion in the mean flow (which, after all, is responsible for the instability) is reduced too much by the action of viscosity. It seemed that this might be a little difficult in the context of axisymmetric flows with disturbances of a given frequency growing or decaying with downstream distance, but I thought that the problem would be basically the same when put in terms of two-dimensional flows with disturbances of given wave-number growing or decaying with time, the latter problem being simpler mathematically. Thus I was led to consider the stability of the flows which are slight distortions of plane Couette flow.

At first I chose a profile for which the distortion had a Gaussian distribution, this corresponding, say, to a wake produced by a flat ribbon placed across the flow. Such a profile has the form

$$U = y - A\exp\{-[(y-y_0)/\epsilon]^2\} \quad (|y| \leqslant 1),$$

$U$ being the velocity in the $x$-direction, and $A$, $y_0$, $\epsilon$ being constants. The first problem is to find the 'threshold' amplitude, $A_{\text{thresh}}$, for which the profile first becomes unstable, given $y_0$ and $\epsilon$. One would expect that long wavelengths would become unstable first, so one looks for the value of $A$ for which long-wavelength disturbances are just neutral. The condition for this is that

$$0 = K_1(c_s) = \int_{-1}^{1} \frac{dy}{(U-c_s)^2} = \left[\frac{-1}{(U-c_s)\,U'}\right]_{-1}^{1} - \int_{-1}^{1} \frac{U''\,dy}{(U-c_s)\,U'^2}, \tag{2.1}$$

where $c_s$ is chosen as the value of $U$ corresponding to the point where $|U'|$ is a maximum. With the help of some subroutines developed by Professors Landahl and Howard, I computed $K_1(c_s)$ for various values of $A$, $y_0$ and $\epsilon$ at the Computation Center at M.I.T. However, it soon became apparent that the threshold amplitude, $A_{\text{thresh.}}$, corresponding to $K_1(c_s)$ being zero could be made as small as desired, simply by taking $\epsilon$ small enough.

This led to a new, and more general, formulation being an asymptotic approach in the limit as $\epsilon \to 0$. The profile is taken to have the form

$$U = y + \epsilon^2 a W[(y-y_0)/\epsilon] \quad (|y| \leqslant 1), \tag{2.2}$$

where $\max_y |W| = 1$ and $|y_0| < 1$. The factor $\epsilon^2$ is one that comes out in the wash so that the value of $a$ for which the profile is just unstable is of order unity.

Differentiating (2.2), we find that the vorticity, $U'$, is given by

$$U' = 1 + \epsilon a W'[(y-y_0)/\epsilon]. \tag{2.3}$$

Thus for a profile for which $W' \to 0$ as $(y-y_0)/\epsilon \to \infty$, the vorticity is uniform except for some additional vorticity near the point $y = y_0$. If the flow is to be unstable, it is necessary for $U'$ to have a local maximum. Introducing the co-ordinate

$$\eta = (y-y_0)/\epsilon \tag{2.4}$$

to designate points in the narrow region of extra vorticity, the vorticity maximum will be at a point $\eta = \eta_s$ given by

$$W''(\eta_s) = 0,$$

with $W''(\eta)/(\eta-\eta_s)$ *negative* on either side of $\eta_s$.

The relation (2.1) for the threshold amplitude becomes, in the limit as $\epsilon \to 0$,

$$0 = K_1(c_s) = -\frac{2}{1-y_0^2} - a\int_{-\infty}^{\infty} \frac{W''(\eta)\,d\eta}{\eta-\eta_s}, \tag{2.5}$$

and it will be assumed that $W(\eta)$ is of such a form that the integral in (2.5) exists and is finite, and that a considerable fraction of the integral on $(-\infty, \infty)$ is contributed by a certain finite interval, $(-1, 1)$ say. Rosenbluth & Simon (1964) have in fact shown that if the profile is monotonic (implied here by (2.3)) and if $W''$ has only a single zero, a *necessary* and *sufficient* condition for instability is that $K_1(c_s) > 0$, that is, in our case that $a > a_{\text{thresh.}}$ where $a_{\text{thresh.}}$ is the threshold value of $a$ given by (2.5).

Notice from (2.2) and (2.3) that although the changes in $U$ and $U'$ are small, the change in $U''$ is of order unity. It is interesting to compare this with Meksyn & Stuart's (1951) result for a rather different problem concerning finite-amplitude disturbances to plane Poiseuille flow, where considerable changes result from small changes in the mean profile, which imply, nevertheless, considerable changes in $U''$.

The next question to be answered is 'What growth rates are associated with a distortion of the mean flow of given amplitude $a$?' It can be seen from (2.5) that for a given distortion profile $W(\eta)$, the threshold amplitude is least when $y_0 = 0$, and becomes relatively large if the narrow region given by $\eta = O(1)$ is close to one of the walls. Thus in the following we will consider only the case $y_0 = 0$, and will further simplify the calculations by assuming that $W(\eta)$ is an antisymmetric function of $\eta$. This means that the vorticity is a symmetric function of $\eta$ with a maximum at the centre. The simplifying feature is that the wave-speed of the disturbance is zero.

The growth rate, $\alpha c_i$, associated with a disturbance of wave-number $\alpha$ can be found in the usual way by looking for self-exciting disturbances whose stream function has the form

$$\operatorname{Re}\{\phi(y)\, e^{i\alpha(x-ct)}\} \tag{2.6}$$

with $c$ the complex number $c = c_r + ic_i$. The equation for $\phi$ expresses the fact that the vorticity of a material particle of fluid remains constant throughout its motion. In the region near the centre where the vorticity $U'$ of the undisturbed flow is not constant, small motions in the $y$-direction tend to give the particle an excess or defect of vorticity relative to its surroundings, this vorticity being transferred to the disturbance. The disturbance vorticity is given by

$$\mathrm{Re}\{(\phi''-\alpha^2\phi)\,e^{i\alpha(x-ct)}\}$$

and the equation representing the transfer of vorticity is

$$(U-c)(\phi''-\alpha^2\phi)-U''\phi = 0. \tag{2.7}$$

In the region away from the centre ($|y/\epsilon| \gg 1$), there is no transfer of vorticity to the disturbance, so that the periodic part of the flow must be irrotational. This is clear for cases in which the vorticity is uniform away from the centre, but may be shown to be true in more general circumstances. In order to satisfy the boundary conditions $\phi(\pm 1) = 0$, we require

$$\phi = \sinh[\alpha(1-|y|)]/\sinh\alpha \quad \text{for} \quad |y/\epsilon| \gg 1, \tag{2.8}$$

where we have adopted the normalization $\phi(0) = 1$. This is the only non-trivial solution that satisfies the boundary condition, is irrotational except at $y = 0$, and has the normal velocity continuous across the surface $y = 0$. However, the solution requires a jump in the tangential velocity of

$$-2\alpha\coth\alpha\,\mathrm{Re}\{e^{i\alpha(x-ct)}\} \tag{2.9}$$

across the surface $y = 0$ representing a vortex sheet with periodically changing strength, and this vorticity has to be supplied by transfer from the mean flow. Sufficient vorticity can be transferred only if the amplitude, $a$, is large enough, i.e. $a > a_{\text{thresh.}}$. On the other hand, given $a > a_{\text{thresh.}}$, the amount of vorticity supplied depends on $c_i$, so the requirement on the amount of vorticity to be supplied gives a relation between $c_i$ and $\alpha$. From (2.7) we have that near $y = 0$

$$\frac{d^2\phi}{dy^2}-\alpha^2\phi = \frac{aW''(\eta)}{\epsilon(\eta-ik)}\phi(0),$$

where $c = i\epsilon k$, so that if $\alpha$ is of order unity the change in $d\phi/dy$ across the layer is given by

$$\Delta\phi_y = a\int_{-\infty}^{\infty}\frac{\eta W''(\eta)\,d\eta}{\eta^2+k^2} = -2\alpha\coth\alpha, \tag{2.10}$$

the latter equality coming from comparison with (2.9). From (2.10) it follows that the range $0 < \alpha < \alpha_s$ of wave-numbers is unstable, where $\alpha_s$ is given by

$$2\alpha_s\coth\alpha_s = -a\int_{-\infty}^{\infty}\frac{W''(\eta)}{\eta}\,d\eta.$$

*Example* (i)
$$W = \begin{cases}\tfrac{1}{2}(3\eta-\eta^3) & \text{for} \quad |\eta| < 1,\\ \operatorname{sgn}\eta & \text{for} \quad |\eta| > 1.\end{cases}$$

This is a profile corresponding to extra vorticity given by

$$W' = \tfrac{3}{2}(1-\eta^2), \quad |\eta| < 1.$$

The relation (2.10) in this case is

$$\alpha\coth\alpha = 3a\{1-k\tan^{-1}(1/k)\},$$

so that the threshold amplitude is $\frac{1}{3}$.

*Example* (ii) $$W = \sin\eta.$$

This example is a little different in that the additional vorticity is not confined to a thin layer, but varies rapidly across the flow. There are several maxima of vorticity, and so several possible solutions, but the one with the smallest threshold amplitude is the solution associated with the maximum at the centre, as is shown by (2.5). The formula (2.8) to (2.10), it turns out, are still valid first approximations in this case since most of the vorticity transfer takes place near the centre where the convection in the $x$-direction is weakest. In particular, (2.10) becomes

$$2\alpha \coth\alpha = \pi a \exp(-k), \tag{2.11}$$

so that the threshold amplitude is $a_{\text{thresh.}} = 2/\pi$. If $a$ is about 1·28 times the threshold amplitude, the maximum value of $\alpha k$ is 0·083 corresponding to the wave-number $\alpha = 0{\cdot}53$, and if $a \approx 2{\cdot}72\, a_{\text{thresh.}}$, the maximum value of $\alpha k$ is 0·76 corresponding to the wave-number $\alpha = 1{\cdot}25$.

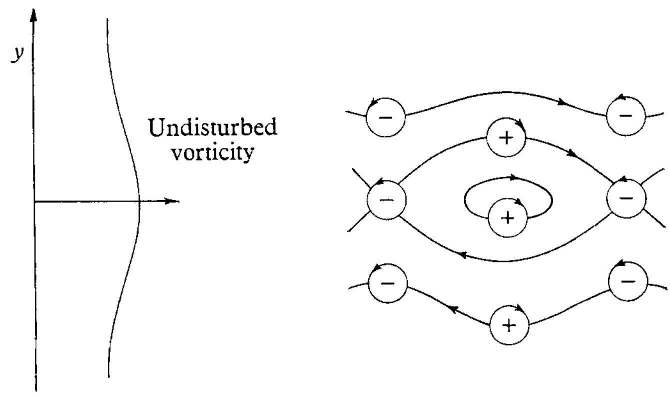


FIGURE 1. On the left is shown an undisturbed vorticity distribution with maximum at $y = 0$. On the right is shown a convecting field of the form normally considered in stability analysis. The circles show the sign of the vorticity acquired by the disturbance under this convection field. It will be observed that the sign of the acquired vorticity is everywhere such as to strengthen the disturbance. If the undisturbed vorticity distribution had a minimum at $y = 0$ instead, the sign of the acquired disturbance vorticity would be reversed and the disturbance inhibited.

In previous formulae such as (2.5) and (2.10) the importance of the sign of $W''(\eta)$ is brought out, that is, the fact that for instability it is necessary to have a local maximum in the modulus of the vorticity. It seems worthwhile examining the physical reason for this. The argument of Lin (1955, §4.4) and Lighthill (Rosenhead 1963, p. 92) cannot differentiate between a maximum and a minimum of vorticity because they only take into account the effects of moving material elements in the $y$-direction. To see why there must be a local maximum of vorticity for instability it is necessary to consider convection of elements in the $x$-direction as well. Now, in a normal-mode approach one looks for disturbances that excite themselves. In other words the disturbed motion will transfer vorticity to the disturbance in such a way that the motion induced by the acquired vorticity will be a strengthening of the original disturbance motion.

Let us look for such a self-exciting disturbance. We will assume that the disturbance has the form (2.5) so that the convecting field will have the form shown in figure 1 with the cat's-eye pattern exhibited in part of the flow. For $y > 0$, a material element displaced upward will have positive vorticity relative to its surroundings, and if displaced downward, negative vorticity. The signs of the acquired disturbance vorticity produced by the convection field are shown in the diagram. It will be observed that the motion induced by this acquired vorticity is everywhere such as to strengthen the disturbance so it is possible for the disturbance to be self-excited. If the undisturbed vorticity distribution had a minimum at $y = 0$ instead of a maximum, the signs of the acquired disturbance vorticity would everywhere be reversed, so the disturbance would be inhibited. If the variation of vorticity were monotonic across $y = 0$, the field could not be self-excited either as the figure would be reversed on one side of the cat's eyes.

The above argument seems to show that it is necessary for instability to have a maximum in the undisturbed vorticity distribution. However, given such a distribution of vorticity, one cannot infer that the disturbance will be self-excited because the boundary conditions have not been taken into account. The previous analysis indicates that the presence of boundaries has an inhibiting effect on the disturbance, which is more pronounced the greater is $\alpha$ because of the greater vorticity required in the central layer to maintain the flow outside the layer.

## 3. Viscous effects and a 'critical' Reynolds number

The instabilities discussed in § 2 are produced by distortions of the mean flow, but such distortions will be damped out by viscosity so long as the Reynolds stresses produced by the disturbance are small enough. Given a change $u(y, t)$ in the mean velocity from the original Couette flow, one finds, on neglecting squares of $u$, that $u$ satisfies the heat equation

$$\partial u/\partial t = R^{-1}\, \partial^2 u/\partial y^2, \tag{3.1}$$

where $R$ is the Reynolds number based on half the difference between the velocities of the plates and half the distance between the plates. There is no $v$-component by continuity. It follows immediately that for changes with a $y$-scale of order $\epsilon$, the time scale for the viscous processes is of order $\epsilon^2 R$.

This is also the time scale on which viscous damping of the periodic disturbances discussed in § 2 would become important. It has been tacitly assumed in § 2 that this time scale is large compared with the time scale for growth under non-viscous processes, the latter being of order $1/\epsilon$ when the amplitude of the distortion of the mean flow is of the same order as the threshold amplitude. In other words, it has been assumed that $\epsilon^3 R$ is large. It will now be shown that $\epsilon^3 R$ must necessarily be large if non-linear effects are to become important, as they must do if the flow is to become turbulent.

The non-linear effects, such as the generation of harmonics and the transfer of energy from the mean flow to the disturbance by the action of Reynolds stresses, become important when the periodic disturbance has grown comparable with the distortion of the mean flow. We will assume that this requires growth by a factor of $(1/\epsilon)^p$, that is that the periodic disturbance is initially smaller than the distortion of the mean flow by a factor $\epsilon^p$, where $p$ is a positive constant. By § 2, the time taken for growth by this amount will be of order $\epsilon^{-1}\log(1/\epsilon)$. During this time the distortion of the mean flow must not have been reduced in amplitude below the threshold amplitude, for otherwise the disturbance would have grown to a maximum and begun to decay before any significant non-linear effects could occur. Since the time scale for changes in the mean flow is of order $\epsilon^2 R$, the Reynolds number must be of order

$$R \sim \epsilon^{-3}\log(1/\epsilon), \tag{3.2}$$

or larger. This is the result that indicates that very large Reynolds numbers are required for instabilities of the type discussed here.

*Example.* The separable solutions of (3.1), antisymmetric in $y$, are given by

$$u = a_0 \sin(n\pi y)\exp(-n^2\pi^2 t/R), \tag{3.3}$$

where $n$ is a positive integer. If $n\pi$ is large, we can identify $1/n\pi$ with $\epsilon$ and the growth rate, $\epsilon\alpha k$, at any given time is given by (2.11), $a$ now being the function of time given by (3.3), namely

$$a = a_0 \exp(-t/\epsilon^2 R);$$

$a_0$ is the value of $a$ at $t = 0$. A 'quasi-linear' theory is valid since the changes in the mean flow are assumed to take place in a time large compared with the time scale for the growth of disturbances. The growth of the disturbance in time $t$ will be by a factor

$$\exp\left[\int_0^t \epsilon\alpha k(t)\,dt\right] = \exp\left(\epsilon\alpha k_0 t - \frac{\alpha t^2}{2\epsilon R}\right),$$

which will be equal to $(1/\epsilon)^p$ when the non-linear effects become important. Assuming that the disturbance is still growing at this stage, say at half the original rate, we have also

$$\epsilon\alpha k_0 - \alpha t/\epsilon R = \tfrac{1}{2}\epsilon\alpha k_0,$$

and eliminating $t$ from the two relationships we obtain

$$R = (8p/3\alpha k_0^2)\,\epsilon^{-3}\log(1/\epsilon).$$

The coefficient $8p/3\alpha k_0^2$ depends on the initial distortion of the mean flow, the initial amplitude of the periodic disturbance, and on the wave number $\alpha$. If $p$ and $a_0$ are given, the coefficient depends on $\alpha$ and has a minimum for a certain value of $\alpha$; e.g. if $p = 2$, $a_0 = 2a_{\text{thresh.}}$, then the minimum value of the coefficient is about 27 corresponding to a wave number of about 0·7. If $n = 1$, $\epsilon = 1/\pi$ (not really small), this gives a 'critical' Reynolds number of about 1000, while if $n = 2$, the 'critical' Reynolds number is more like 10,000. Of course these figures only give a rough idea of the Reynolds numbers involved as they are based on somewhat arbitrary standards. If a larger initial amplitude of the distortion in the mean flow is allowed then, of course, the 'critical' Reynolds number will be smaller. The 'critical' Reynolds number is also smaller if the periodic disturbances are initially greater in magnitude.

## 4. The case of Poiseuille flow in a pipe

Unless the layer in which $d(r^{-1}\, dU/dr)/dr$ is non-zero is very close to the centre of the pipe, the behaviour of the periodic disturbance in the layer is, to the first order, the same as in the two-dimensional case since the layer is thin compared with its radius. $r$ is the radial co-ordinate. The irrotational flow outside the layer will, of course, be different in the axisymmetric case, so that, for instance, the expression $\alpha \coth \alpha$ on the right-hand side of (2.10) will be replaced by an expression involving modified Bessel functions. Also it does not make a great deal of difference whether one considers disturbances growing in space or growing in time because the growth rate is small (cf. Gaster 1962).

For the axisymmetric case there are various ways of defining $\epsilon$. Here we will define $\epsilon$ as the change in $r^2$ across the layer so that $\epsilon$ is a measure of the cross-sectional area of the layer. This definition is convenient in the discussion of the threshold amplitude because, for $\alpha = 0$, the equation for axisymmetric disturbances to an axisymmetric flow is the same as (2.7) with $\alpha = 0$ and with $y = r^2$ (Batchelor & Gill 1962, equation (2.16) with $n = 0$ and $rG = \phi$). The boundary conditions are $\phi = 0$ at $y = 0$ and $y = 1$ and the equilibrium profile is $U = 1-y$ so that the problem is exactly the same as for plane Couette flow. The threshold amplitude will be smallest when $y = \frac{1}{2}(r \approx 0{\cdot}7)$ and becomes relatively large when the layer is near the wall or near the centre of the pipe.

The theory as developed so far seems to fit the conditions of Leite's (1956) experiment very satisfactorily, but what about the situation of Reynolds's original experiment? Here fluid enters the pipe with approximately uniform velocity setting up at the wall a vortex sheet which diffuses in toward the centre, the Poiseuille profile eventually being set up (Goldstein 1938, § 139). There seem to be no experimental results available which show the changes in mean profile and disturbance amplitude in detail, so we can only speculate on possible causes of the observed instability. Tatsumi (1952) has shown that the boundary layer is unstable in part of the entry region, but it is not clear whether this instability plays an important role in producing the observed effects such as the sudden rapid oscillations of the dye column in Reynolds's experiment at a certain distance from the nozzle.

However, it is instructive to consider the stability of slightly distorted profiles. Before the vortex layer at the wall has diffused to the radius at which the 'bump' in the mean profile is situated, the bump will just represent a velocity change of order $\epsilon^2$ superimposed on the uniform flow. If the effects of the boundary layer near the wall are ignored, the stability of such profiles for wave-numbers of order unity can be examined by a method similar to that of Drazin & Howard (1962) since the wavelength is large compared with the thickness of the layer. The growth rate is of order $\epsilon^2$ for a half-jet profile and of order $\epsilon^{\frac{3}{2}}$ for a full-jet profile. In either case the rate of growth of disturbances is very small until vorticity has diffused out as far as the layer and the effects of the shear become felt. Then the growth rate will be of order $\epsilon$ as was found in § 2. This sudden change in growth rate could well be associated with the delay in noticeable instabilities until a certain distance from the nozzle. On the other hand, in the distance which is required for the vorticity to diffuse out to the region of the layer, the bump on the mean-flow profile will have diffused considerably also, unless Reynolds stresses due to periodic disturbances are sufficient to counteract the diffusion to some extent at least. It is to be hoped that experiments will help to throw light on this question.

Finally, it should be remarked that the mechanism of instability discussed in this paper is a finite amplitude one, but differs, from the non-linear model for instability of plane Couette flow suggested by Watson (1960). In Watson's model changes in the mean flow are secondary in the sense that they are produced by Reynolds stresses resulting from a fluctuating disturbance. The model discussed in this paper on the other hand, suggests that the zero wave-number component of the disturbance spectrum, that is the *imposed* change in the mean flow, plays an important part in the instability.

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## REFERENCES

Batchelor, G. K. & Gill, A. E. 1962 *J. Fluid Mech.* **14**, 529.

Drazin, P. G. & Howard, L. N. 1962 *J. Fluid Mech.* **14**, 257.

Gaster, M. 1962 *J. Fluid Mech.* **14**, 222.

Gill, A. E. 1965 *J. Fluid Mech.* **21**, 145.

Goldstein, S. 1938 *Modern Developments in Fluid Mechanics.* Oxford: Clarendon Press.

Kuethe, A. M. 1956 *J. Aero. Sci.* **23**, 446.

Leite, R. J. 1956 *Univ. Michigan Engng Coll. Rep.* IP-188.

Lin, C. C. 1955 *The Theory of Hydrodynamic Stability.* Cambridge University Press.

Meksyn, D. & Stuart, J. T. 1951 *Proc. Roy. Soc.* A, **208**, 517.

Rosenbluth, M. N. & Simon, A. 1964 *Phys. Fluids*, **7**, 557.

Rosenhead, L. 1963 *Laminar Boundary Layers.* Oxford: Clarendon Press.

Tatsumi, T. 1952 *J. Phys. Soc. Japan*, **7**, 489.

Watson, J. 1960 *J. Fluid Mech.* **9**, 371.